Journal of Applied Mechanics and Technical Physics, Vol. 35, No. 1, 1994

PLANAR DEFORMATION IN GEOMETRICALLY NONLINEAR ELASTICITY

V. D. Bondar'

UDC 539.3

We will consider planar deformation within the framework of Novozhilov's variant of nonlinear elasticity. Equations in stresses and strains will be derived, and their elliptical type established. It will be shown that in this variant, as in linear elasticity, the characteristics of deformation are representable by two complex potentials, while the planar elasticity problem itself reduces to a boundary problem for the potentials. However, in contrast to linear theory the representations referred to and the boundary problem become nonlinear. The dynamic condition for which linear theory follows from nonlinear is stated. An analytical solution is presented for one of the problems, and a general method developed for solution of the nonlinear boundary problem.

1. In the study of many practical mechanical problems linear elasticity theory does not provide the needed accuracy, so is replaced by nonlinear theory. Nonlinearity may be present in both the law describing mechanical behavior of the material (physical nonlinearity) and special features of the deformation (geometric nonlinearity). Among the latter cases is the nonlinear theory of elasticity developed by V. V. Novozhilov [1]. In that theory it is assumed that rotations and extension-shear of material elements are small in comparison to unity and that the former may significantly exceed the latter. Such a situation is realized in a number of cases, in particular, deformation of flexible bodies: bars, plates, and shells. The assumptions made permit simplification of the general nonlinear representation of deformations in terms of extension-shears and rotations and use of a special nonlinear formulation.

In Novozhilov's variant of elasticity the static problem is described by equilibrium equations, Hook's law, and a special nonlinear relationship between deformations and rotations and extension-shears, representations of the latter in terms of displacements, and boundary conditions specifying displacements on one portion Σ_u of the deformed body surface and stresses on another portion Σ_p:

$$\begin{gathered} \operatorname{div} P + \mathbf{f} = 0, \\ P = \lambda \varepsilon_1 G + 2\mu\varepsilon, \quad \varepsilon_1 = \operatorname{tr} \varepsilon, \quad \lambda = \text{const}, \quad \mu = \text{const}, \\ 2\varepsilon = 2e + \omega\cdot\omega, \quad 2e = \nabla \mathbf{u} + (\nabla \mathbf{u})^*, \quad 2\omega = \nabla \mathbf{u} - (\nabla \mathbf{u})^*; \end{gathered} \tag{1.1}$$

$$\mathbf{u}|_{\Sigma_u} = \mathbf{h}, \quad P\cdot\mathbf{n}|_{\Sigma_p} = \mathbf{p}. \tag{1.2}$$

Here λ, μ are elasticity coefficients, **u**, **f**, **h**, **p**, **n** are displacement, volume force density, boundary displacement, stress, and external normal vectors; G, P, ε, e, ω, ∇u, (∇u)* are the metric tensor and tensors describing stress, deformation, elongation-shear, rotation, the gradient, and transposed gradient of displacement.

For planar deformation of a cylindrical body the planar fundamental problem is that in which Eq. (1.1) is satisfied in a planar region S, while Eq. (1.2) is satisfied on that region's boundary L. In Cartesian coordinates x, y for the deformed state Eqs. (1.1), (1.2) (for potential forces and energy V) have the form

$$\begin{gathered} \frac{\partial (P_{xx} - V)}{\partial x} + \frac{\partial P_{xy}}{\partial y} = 0, \quad \frac{\partial P_{xy}}{\partial x} + \frac{\partial (P_{yy} - V)}{\partial y} = 0, \\ P_{xx} = \lambda (\varepsilon_{xx} + \varepsilon_{yy}) + 2\mu\varepsilon_{xx}, \quad P_{yy} = \lambda (\varepsilon_{xx} + \varepsilon_{yy}) + 2\mu\varepsilon_{yy}, \quad P_{xy} = 2\mu\varepsilon_{xy}, \\ 2\varepsilon_{xx} = 2e_{xx} - \omega_{xy}^2, \quad 2\varepsilon_{yy} = 2e_{yy} - \omega_{xy}^2, \quad \varepsilon_{xy} = e_{xy}, \end{gathered} \tag{1.3}$$

Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 1, pp. 99-114, January-February, Original article submitted April 26, 1993

0021-8944/94/3501-0100$12.50

$$e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{yy} = \frac{\partial u_y}{\partial y}, \quad 2e_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}, \quad 2\omega_{xy} = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y};$$
$$u_x|_{L_u} = h_x(s), \quad u_y|_{L_u} = h_y(s),$$
$$P_{xx}\frac{dy}{ds} - P_{xy}\frac{dx}{ds}\bigg|_{L_p} = p_x(s), \quad P_{yx}\frac{dy}{ds} - P_{yy}\frac{dx}{ds}\bigg|_{L_p} = p_y(s), \tag{1.4}$$

where L_u, L_p are portions of the boundary L on which displacements and stresses, respectively, are specified; the Cartesian components of the vectors and tensors are denoted by the same symbols as the quantities themselves, but with literal subscripts; moreover, it is considered that the components of the normal are defined by the equation of the contour Lx = x(s), y = y(s) (where s is an arc of L) by the expressions $n_x = dy/ds$, $n_y = -dx/ds$.

Planar deformation can also be described in the complex coordinates $z = x + iy$, $\bar{z} = x - iy$ (considered as independent variables), in which system (1.3), (1.4) becomes more compact. To do this we transform to differentiation with respect to the complex variables using the expressions

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

and make use, for example, of contravariant complex components of the vectors and tensors (denoted by the previous symbols, now with numerical superscripts), which are related to the Cartesian components of the corresponding quantities by the usual expressions for component transformation, having the forms

$$u^1 = \overline{u^2} = u = u_x + iu_y,$$
$$h^1 = \overline{h^2} = h = h_x + ih_y, \quad p^1 = \overline{p^2} = p = p_x + ip_y,$$
$$P^{11} = \overline{P^{22}} = P_{xx} - P_{yy} + 2iP_{xy}, \quad P^{12} = P_{xx} + P_{yy},$$
$$\varepsilon^{11} = \overline{\varepsilon^{22}} = \varepsilon_{xx} - \varepsilon_{yy} + 2i\varepsilon_{xy}, \quad \varepsilon^{12} = \varepsilon_{xx} + \varepsilon_{yy},$$
$$e^{11} = \overline{e^{22}} = e_{xx} - e_{yy} + 2ie_{xy}, \quad e^{12} = e_{xx} + e_{yy},$$
$$\omega^{11} = \overline{\omega^{22}} = 0, \quad \omega^{21} = \overline{\omega^{12}} = 2i\omega_{xy}.$$

In the final outcome Eqs. (1.3), (1.4) in complex coordinates appear as

$$\frac{\partial P^{11}}{\partial z} + \frac{\partial (P^{12} - 2V)}{\partial \bar{z}} = 0, \quad P^{11} = \overline{P^{22}} = 2\mu\varepsilon^{11}, \quad P^{12} = 2(\lambda + \mu)\varepsilon^{12}, \tag{1.5}$$
$$\varepsilon^{11} = \overline{\varepsilon^{22}} = e^{11}, \quad \varepsilon^{12} = e^{12} + \frac{1}{4}(\omega^{21})^2,$$
$$e^{11} = \overline{e^{22}} = 2\frac{\partial u}{\partial \bar{z}}, \quad e^{12} = \frac{\partial u}{\partial z} + \frac{\partial \bar{u}}{\partial \bar{z}}, \quad \omega^{21} = \frac{\partial u}{\partial z} - \frac{\partial \bar{u}}{\partial \bar{z}};$$
$$u|_{L_u} = h(s), \quad P^{12}\frac{dz}{ds} - P^{11}\frac{d\bar{z}}{ds}\bigg|_{L_p} = 2ip(s). \tag{1.6}$$

2. Let certain displacements be specified on the boundary of the planar region (L_u = L), so that it will be convenient to solve the planar problem in displacements. The problem in displacements in Cartesian coordinates follows from Eqs. (1.3), (1.4), after elimination of the unknowns:

$$\Phi_1 = 2(1 - \nu)\frac{\partial^2 u_x}{\partial x^2} + \omega_{xy}\frac{\partial^2 u_x}{\partial x \partial y} + (1 - 2\nu)\frac{\partial^2 u_x}{\partial y^2} - \omega_{xy}\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} - \frac{1 - 2\nu}{\mu}\frac{\partial V}{\partial x} = 0, \tag{2.1}$$
$$\Phi_2 = 2(1 - \nu)\frac{\partial^2 u_y}{\partial y^2} - \omega_{xy}\frac{\partial^2 u_y}{\partial x \partial y} + (1 - 2\nu)\frac{\partial^2 u_y}{\partial x^2} + \omega_{xy}\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial x \partial y} - \frac{1 - 2\nu}{\mu}\frac{\partial V}{\partial y} = 0;$$
$$u_x|_L = h_x(s), \quad u_y|_L = h_y(s). \tag{2.2}$$

Here ω_{xy} is defined in terms of displacements by Eq. (1.3); $\nu = \lambda/(2(\lambda + \mu))$ is the Poisson coefficient. Quasilinear system (2.1) is a generalization of the Lamé equations of linear elasticity.

To study the type of Eq. (2.1), we take $x_1 = x$, $x_2 = y$, $w_1 = u_x$, $w_2 = u_y$, and, following [2], construct the characteristic determinant

$$\Delta_* = \det(A_{\alpha\beta}), \quad A_{\alpha\beta} = \sum_{k_1+k_2=2} \frac{\partial \Phi_\alpha}{\partial \left\{ \dfrac{\partial^2 w_\beta}{\partial x_1^{k_1} \partial x_2^{k_2}} \right\}} \sigma_1^{k_1}\sigma_2^{k_2} \qquad (2.3)$$

$$(\alpha, \beta = 1, 2).$$

According to Eqs. (2.1), (2.3) the elements of the determinant and its magnitude have the form

$$A_{11} = 2(1-\nu)\sigma_1^2 + \omega_{xy}\sigma_1\sigma_2 + (1-2\nu)\sigma_2^2, \quad A_{12} = -\omega_{xy}\sigma_1^2 + \sigma_1\sigma_2,$$
$$A_{22} = 2(1-\nu)\sigma_2^2 - \omega_{xy}\sigma_1\sigma_2 + (1-2\nu)\sigma_1^2, \quad A_{21} = \omega_{xy}\sigma_2^2 + \sigma_1\sigma_2,$$
$$\Delta_* = A_{11}A_{22} - A_{12}A_{21} = 2(1-\nu)(1-2\nu)(\sigma_1^2 + \sigma_2^2)^2.$$

Since the Poisson coefficient varies within the interval $0 < \nu < 0.5$, then $\Delta_* > 0$. Consequently, the characteristic equation $\Delta_* = 0$ has no real roots. Thus, quasilinear system (2.1), like the Lamé linear elasticity system, is elliptical and boundary problem (2.1), (2.2) is correct.

The problem in displacements in the complex variables can be obtained from Eqs. (1.5), (1.6) in an analogous manner and has the form

$$2(1-2\nu)\frac{\partial^2 u}{\partial z \partial \bar{z}} + \frac{\partial}{\partial \bar{z}}\left[\frac{\partial u}{\partial z} + \frac{\partial \bar{u}}{\partial \bar{z}} + \frac{1}{4}\left(\frac{\partial u}{\partial z} - \frac{\partial \bar{u}}{\partial \bar{z}}\right)^2 - \frac{1-2\nu}{\mu}V\right] = 0; \qquad (2.4)$$

$$u|_L = h(s). \qquad (2.5)$$

Equation (2.4) allows complete integration. In fact, after integration over $\bar{z}$ it reduces to an expression containing the arbitrary function $\varphi'(z)$:

$$(3-4\nu)\frac{\partial u}{\partial z} + \frac{\partial \bar{u}}{\partial \bar{z}} + \frac{1}{4}\left(\frac{\partial u}{\partial z} - \frac{\partial \bar{u}}{\partial \bar{z}}\right)^2 - \frac{1-2\nu}{\mu}V = 4\frac{(1-\nu)(1-2\nu)}{\mu}\varphi'(z).$$

Addition and subtraction of this expression to and from its complex conjugate yields the relationships

$$\mu\left(\frac{\partial u}{\partial z} - \frac{\partial \bar{u}}{\partial \bar{z}}\right) = 2(1-\nu)[\varphi'(z) - \overline{\varphi'(z)}],$$
$$\mu\left(\frac{\partial u}{\partial z} + \frac{\partial \bar{u}}{\partial \bar{z}}\right) = (1-2\nu)[\varphi'(z) + \overline{\varphi'(z)}] - \frac{1-\nu}{2\mu}[\varphi'(z) - \overline{\varphi'(z)}]^2 + \frac{1-2\nu}{2(1-\nu)}V,$$

defining the derivative $\partial u/\partial z$ in the form

$$2\mu\frac{\partial u}{\partial z} = \varkappa\varphi'(z) - \overline{\varphi'(z)} - \alpha[\varphi'(z) - \overline{\varphi'(z)}]^2 + \frac{c}{2}V,$$
$$\varkappa = 3 - 4\nu, \quad \alpha = \frac{1-\nu}{2\mu}, \quad c = \frac{1-2\nu}{1-\nu}. \qquad (2.6)$$

This equality can be represented as the inhomogeneous equation

$$\frac{\partial}{\partial z}\left\{2\mu u - \varkappa\varphi(z) + z\overline{\varphi'(z)} + \alpha\left[z\bar{\varphi}'^2(\bar{z}) - 2\varphi(z)\overline{\varphi'(z)} + \int \varphi'^2(z)\,dz\right]\right\} = \frac{c}{2}V,$$

the general solution of which will be [3]

$$2\mu u = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} + \frac{c}{2}W(z,\bar z) - \\ - \alpha\,[z\bar\varphi'^2(\bar z) - 2\varphi(z)\overline{\varphi'(z)} + \int \varphi'^2(z)\,dz], \tag{2.7}$$

where $\psi(z)$ is an arbitrary function and the quantity W for a piecewise-smooth contour L and Helder function V is defined by the expression [4]

$$W(z,\bar z) = -\frac{1}{\pi}\iint_S \frac{V(\xi,\eta)\,d\xi d\eta}{\xi - \bar z}, \quad \zeta = \xi + i\eta. \tag{2.8}$$

In the case of an infinite region the potential energy in Eq. (2.8) must have the property

$$V = O(1/|\zeta|^{1+\beta}), \quad \beta > 0 \quad \text{as} \quad |\zeta| \to \infty.$$

For arbitrary functions W there are valid expressions in which the integral is understood in the sense of the Cauchy main value [4]:

$$\frac{\partial W}{\partial z} = V, \quad \frac{\partial W}{\partial \bar z} = -\frac{1}{\pi}\iint_S \frac{V(\xi,\eta)\,d\xi d\eta}{(\xi - \bar z)^2}. \tag{2.9}$$

Equation (2.7) defines the displacement in terms of two analytical functions $\varphi(z)$ and $\psi(z)$ — the complex potentials. Substitution of Eq. (2.7) in Eq. (1.5) leads to representations of other characteristics of the stress–deformed state in terms of the potentials:

$$P^{11} = \overline{P^{22}} = -2\,[z\overline{\varphi''(z)} + \overline{\psi'(z)}] + cW_{\bar z} - 4\alpha\overline{\varphi''(z)}\,[z\overline{\varphi'(z)} - \varphi(z)], \\ P^{12} = 2\,[\varphi'(z) + \overline{\varphi'(z)}] + kW_z + 2\alpha\,[\varphi'(z) - \overline{\varphi'(z)}]^2, \quad k = 1/(1-\nu); \tag{2.10}$$

$$\mu e^{11} = \mu\overline{e^{22}} = -z\overline{\varphi''(z)} - \overline{\psi'(z)} + \frac{c}{2}W_{\bar z} - 2\alpha\overline{\varphi''(z)}\,[z\overline{\varphi'(z)} - \varphi(z)], \\ \mu e^{12} = (1-2\nu)\,[\varphi'(z) + \overline{\varphi'(z)}] + \frac{c}{2}W_z - \alpha\,[\varphi'(z) - \overline{\varphi'(z)}]^2, \\ \mu\omega^{21} = 2(1-\nu)\,[\varphi'(z) - \overline{\varphi'(z)}], \tag{2.11}$$

while substitution in the condition on the contour, Eq. (2.5), yields a boundary problem for the potentials themselves:

$$\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \alpha\,[z\bar\varphi'^2(\bar z) - 2\varphi(z)\overline{\varphi'(z)} + \\ + \int \varphi'^2(z)\,dz]|_L = h_0(s), \quad h_0(s) = 2\mu h(s) - \frac{c}{2}W(s). \tag{2.12}$$

Thus, in the Novozhilov variant of nonlinear elasticity (as in linear theory) displacements, stresses, deformations, elongation-shears, and rotations can be represented in terms of complex potentials defined by a boundary problem. Expressions (2.7)-(2.12) are nonlinear in the potentials. Their structure is such that together with the nonlinear terms they also contain linear terms coinciding with the corresponding expressions of Kolosov's linear elasticity [5]; thus, the expressions constructed here are a generalization of Kolosov's equations.

From the form of boundary condition (2.12) we conclude that the terms generated by volume forces do not contain the potentials, so that the presence of volume forces does not change the type of the boundary problem.

3. We will assume that some stresses are specified on the entire boundary of the region (L_p = L), and consider the planar problem in stresses and rotations.

Equation (1.5) for the stress and rotation components permits expressing the displacement gradients as

$$2\mu \frac{\partial u}{\partial z} = \frac{1-2\nu}{2} P^{12} + \mu\omega^{21}\left(1 - \frac{\omega^{21}}{4}\right), \quad 2\mu \frac{\partial u}{\partial \bar{z}} = \frac{1}{2} P^{11}. \tag{3.1}$$

It can easily be seen that the consistency condition for this system yields an equation for consistency of stresses and rotations. The latter together with equilibrium equation (1.5) forms a closed quasilinear system of two first order complex equations for the complex and real stresses P^{11}, P^{12}, and the purely imaginary rotation ω^{21}; combination with the force boundary condition equations (1.6) then yields a boundary problem for the stresses and rotation

$$\frac{\partial P^{11}}{\partial z} - \frac{\partial}{\partial \bar{z}}\left[(1-2\nu) P^{12} + 2\mu\omega^{21}\left(1 - \frac{1}{4}\omega^{21}\right)\right] = 0,$$
$$\frac{\partial P^{11}}{\partial z} + \frac{\partial (P^{12} - 2V)}{\partial \bar{z}} = 0; \tag{3.2}$$

$$P^{12}\frac{dz}{ds} - P^{11}\frac{d\bar{z}}{ds}\bigg|_L = 2lp(s). \tag{3.3}$$

System (3.2) is also fully integrable. In fact, elimination of the stress P^{11} leads to the expression

$$\frac{\partial}{\partial \bar{z}}\left[P^{12} - kV + \frac{1}{2\alpha}\omega^{21}\left(1 - \frac{1}{4}\omega^{21}\right)\right] = 0,$$

which after integration takes on the form

$$P^{12} - kV + \frac{1}{2\alpha}\omega^{21}\left(1 - \frac{1}{4}\omega^{21}\right) = 4\varphi'(z), \tag{3.4}$$

where $\varphi'(z)$, an arbitrary function, is the complex potential, and the parameters k and α are defined by Eqs. (2.6) and (2.10). Separating the real and imaginary components in Eq. (3.4), we obtain

$$P^{12} - kV - \frac{1}{8\alpha}(\omega^{21})^2 = 2\,[\varphi'(z) + \overline{\varphi'(z)}], \quad \frac{1}{2\alpha}\omega^{21} = 2\,[\varphi'(z) - \overline{\varphi'(z)}],$$

which determine the real stress and the purely imaginary rotation as functions of the complex potential

$$P^{12} = 2\,[\varphi'(z) + \overline{\varphi'(z)}] + 2\alpha\,[\varphi'(z) - \overline{\varphi'(z)}]^2 + kV; \tag{3.5}$$

$$\mu\omega^{21} = 2\,(1-\nu)\,[\varphi'(z) - \overline{\varphi'(z)}]. \tag{3.6}$$

With consideration of Eqs. (2.9) and (3.5) equilibrium equation (3.2) becomes an equation for the complex stress P^{11}:

$$\frac{\partial}{\partial \bar{z}}\{P^{11} + 2z\overline{\varphi''(z)} + 4\alpha\overline{\varphi''(z)}\,[z\overline{\varphi'(z)} - \varphi(z)] - cW_{\bar{z}}\} = 0$$

defining the latter after integration in the form

$$P^{11} = -2\,[z\overline{\varphi''(z)} + \overline{\psi'(z)}] - 4\alpha\overline{\varphi''(z)}\,[z\overline{\varphi'(z)} - \varphi(z)] + cW_{\bar{z}}, \tag{3.7}$$

where $\psi'(z)$ is an arbitrary function; the parameter c and function W are defined by Eqs. (2.6) and (2.8). Equations (3.5)-(3.7) coincide with Eqs. (2.10) and (2.11) for stresses and rotation, obtained previously by another method.

With consideration of Eqs. (3.5) and (3.7) boundary condition (3.3) can be represented as

$$\frac{d}{ds}\Big\{\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} - \frac{c}{2}W(z,\bar z) +$$
$$+ \alpha\,[z\bar\varphi'^2(\bar z) - 2\varphi(z)\overline{\varphi'(z)} + \int \varphi'^2(z)\,dz]\Big\}\Big|_L = ip - V\frac{dz}{ds}$$

which after integration along the contour yields

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \alpha\,[z\bar\varphi'^2(\bar z) - 2\varphi(z)\overline{\varphi'(z)} + \int \varphi'^2(z)\,dz]|_L = g(s),$$
$$g(s) = \int_0^s \left(ip - V\frac{dz}{ds}\right) ds + \frac{c}{2}W(s) + \text{const}. \tag{3.8}$$

Thus, in solving the problem in stresses and rotations the planar elasticity problem also reduces to a nonlinear boundary problem for complex potentials. This problem is analogous to Eq. (2.12) with specification of boundary displacements, and its type is also independent of the presence of volume forces. The problem of Eq. (3.8) differs from the corresponding problem of linear elasticity [4] only in the nonlinear terms.

Taking the stresses and rotation as defined by Eqs. (3.5)-(3.7) we may consider Eq. (3.1) as equations for displacements. The conditions for consistency of this system are satisfied by the first expression of Eq. (3.2), so that we can write the differential displacements in the form

$$2\mu du = \left[\frac{1-2\nu}{2}P^{12} + \mu\omega^{21}\left(1 - \frac{1}{4}\omega^{21}\right)\right] dz + \frac{1}{2}P^{11}d\bar z =$$
$$= d\left\{\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \alpha\,[z\bar\varphi'^2(\bar z) - 2\varphi(z)\overline{\varphi'(z)} + \int \varphi'^2(z)\,dz] + \frac{c}{2}W(z,\bar z)\right\}.$$

Hence, by integration we establish that the displacement itself is given in terms of the potentials by

$$2\mu u = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} + \frac{c}{2}W(z,\bar z) -$$
$$- \alpha\,[z\bar\varphi'^2(\bar z) - 2\varphi(z)\overline{\varphi'(z)} + \int \varphi'^2(z)\,dz] + \text{const}, \tag{3.9}$$

which differs from Eq. (2.7) only by an insignificant additive constant. Thus, if the potentials are defined by boundary stresses, the displacements are also defined by them to the accuracy of translations of the body as a rigid whole.

One variant of the problem for rotation and stresses, Eqs. (3.2), (3.3), is the problem of rotation and stress functions. If we represent the stresses in terms of the real stress function U with the expressions

$$P^{11} = \overline{P^{22}} = -4\frac{\partial^2 U}{\partial \bar z^2}, \quad P^{12} = 4\frac{\partial^2 U}{\partial z \partial \bar z} + 2V, \tag{3.10}$$

then the equilibrium equation is satisfied identically, while the remaining expressions yield the following boundary problem for the stress functions and rotation:

$$\frac{\partial}{\partial \bar z}\left[4\frac{\partial^2 U}{\partial z\partial \bar z} + cV + \frac{1}{2\alpha}\omega^{21}\left(1 - \frac{1}{4}\omega^{21}\right)\right] = 0; \tag{3.11}$$

$$2\left.\frac{\partial U}{\partial \bar z}\right|_L = 2\left(\frac{\partial U}{\partial \bar z}\right)_0 + \int\limits_0^s \left(ip - V\frac{dz}{ds}\right) ds. \tag{3.12}$$

Here $(U_{\bar z})_0$ is the value of the derivative at s = 0, the parameters α and c are defined by Eq. (2.6).

The problem of Eqs. (3.11), (3.12) can also be reduced to a boundary problem for the complex potentials. In fact, integration of Eq. (3.11) leads to equality of the expressions in square brackets of the arbitrary function $4\varphi'(z)$, and after separation of real and imaginary components yields

$$4\frac{\partial^2 U}{\partial z \partial \bar z} + cV - \frac{1}{8\alpha}(\omega^{21})^2 = 2\,[\varphi'(z) + \overline{\varphi'(z)}], \quad \omega^{21} = 4\alpha\,[\varphi'(z) - \overline{\varphi'(z)}]. \tag{3.13}$$

The expression obtained for the rotation coincides with Eq. (2.11). With consideration of the expression for ω^{21}, as well as the well known [4] representation of the potential energy in terms of complex and real functions, W, K

$$V = \frac{\partial W}{\partial z} = 2\frac{\partial^2 K}{\partial z \partial \bar z}, \quad K = \frac{1}{\pi}\iint\limits_S V(\xi, \eta)\ln|\zeta - z|\, d\xi d\eta \tag{3.14}$$

the first equation of (3.13) becomes an equation for the stress function

$$2\frac{\partial^2 U}{\partial z \partial \bar z} = \varphi'(z) + \overline{\varphi'(z)} - c\frac{\partial^2 K}{\partial z \partial \bar z} + \alpha\,[\varphi'(z) - \overline{\varphi'(z)}]^2,$$

defining the latter in terms of the potentials $\varphi(z)$ and $\psi(z)$ in the form

$$\begin{gathered} 2U = \bar z\varphi(z) + z\overline{\varphi(z)} + \int \psi(z)\,dz + \overline{\int \psi(z)\,dz} - cK(z, \bar z) + \\ + \alpha\,[\bar z \int \varphi'^2(z)\,dz + z\int \bar\varphi'^2(\bar z)\,d\bar z - 2\varphi(z)\overline{\varphi(z)}]. \end{gathered} \tag{3.15}$$

Thus, the stress function is defined in a nonlinear manner in terms of the complex potentials. Note that the portion of Eq. (3.15) linear in the potentials coincides with the stress function expression of linear elasticity [6]. It can easily be seen that the stresses of Eq. (3.10) defined by Eq. (3.15) coincide with the expressions of Eq. (2.10) for these quantities, while the boundary condition (3.12) coincides with condition (3.8).

The generalized displacements, rotations, and elongation-shears of Eqs. (2.7), (2.11) $U^\alpha = \mu u^\alpha$, $\Omega^{\alpha\beta} = \mu\omega^{\alpha\beta}$, $E^{\alpha\beta} = \mu e^{\alpha\beta}$, like the stresses of Eq. (2.10), are defined by one and the same potentials, and are thus finite values. Let l_0 and P_0 be characteristic length and stress, while $\sigma = P_0/\mu$ is the characteristic dimensionless stress. We will compare these quantities to the corresponding dimensionless quantities, denoted by asterisks, using the expressions

$$\begin{gathered} U^\alpha = l_0 P_0 U_*^\alpha, \quad W = l_0 P_0 W_*, \\ P^{\alpha\beta} = P_0 P_*^{\alpha\beta}, \quad p^\alpha = P_0 p_*^\alpha, \quad V = P_0 V_*, \\ U = l_0^2 P_0 U_*, \quad K = l_0^2 P_0 K_*, \quad \Omega^{\alpha\beta} = P_0 \Omega_*^{\alpha\beta}, \\ \varphi' = P_0\varphi_*', \quad \psi' = P_0\psi_*', \quad z = l_0 z_*, \quad s = l_0 s_*. \end{gathered}$$

Then the relationships of the problem for stress functions and rotation, Eqs. (3.10)-(3.12) and the representations of the unknown quantities in terms of the potentials of Eqs. (3.13), (3.14) in dimensionless variables takes on the form

$$\begin{gathered} P_*^{11} = \overline{P_*^{22}} = -4\frac{\partial^2 U_*}{\partial \bar z_*^2}, \quad P_*^{12} = 4\frac{\partial^2 U_*}{\partial z_* \partial \bar z_*} + 2V_*, \\ \frac{\partial}{\partial \bar z_*}\left[4\frac{\partial^2 U_*}{\partial z_* \partial \bar z_*} + cV_* + \frac{\Omega_*^{21}}{1-\nu}\left(1 - \frac{\sigma}{4}\Omega_*^{21}\right)\right] = 0, \end{gathered} \tag{3.16}$$

$$\left.\frac{\partial U_*}{\partial \bar z_*}\right|_L = \left(\frac{\partial U_*}{\partial \bar z_*}\right)_0 + \int_0^{s_*} \left(ip_* - V_* \frac{dz_*}{ds_*}\right) ds_*, \quad \Omega_*^{21} = 2(1-\nu)(\varphi_*' - \bar\varphi_*'),$$

$$U_* = \mathrm{Re}\left[\bar z \varphi_* + \int \psi_* dz_* - \frac{c}{2}K_* + \sigma \frac{1-\nu}{2}\left(\bar z_* \int \varphi_*'^2 dz_* - \varphi_*\bar\varphi_*\right)\right].$$

Assuming that the dimensionless functions are finite in modulus, while the parameter σ is quite small in comparison to unity, in Eq. (3.16) we can neglect terms containing this factor as being small in modulus as compared to other terms. In the final outcome we obtain dimensionless expressions

$$P^{11} = \overline{P^{22}} = -4\frac{\partial^2 U}{\partial \bar z^2}, \quad P^{12} = 4\frac{\partial^2 U}{\partial z \partial \bar z} + 2V,$$

$$\frac{\partial}{\partial \bar z}\left[4\frac{\partial^2 U}{\partial z \partial \bar z} + cV + \frac{\mu\Omega^{21}}{1-\nu}\right] = 0,$$

$$2\left.\frac{\partial U}{\partial \bar z}\right|_L = 2\left(\frac{\partial U}{\partial \bar z}\right)_0 + \int_0^s \left(ip - V\frac{dz}{ds}\right) ds,$$

$$\Omega^{21} = 2(1-\nu)(\varphi' - \bar\varphi'), \quad U = \mathrm{Re}\left(\bar z\varphi + \int \psi dz - \frac{c}{2}K\right),$$

coinciding with the known expressions of linear elasticity [5, 6].

From Eqs. (2.7) and (2.11) we can also establish that in the expressions for the dimensionless quantities U_*^α, $E_*^{\alpha\beta}$, and $\Omega_*^{\alpha\beta}$ for very small σ it is also possible to neglect nonlinear terms containing the factor σ, after which they take on the same form as in linear elasticity. Thus, the results of linear theory follow from the results of the given nonlinear model for very small σ, i.e., for characteristic stresses very small in comparison to the elastic constant of the material.

We will assume that S is a singly connected finite (or infinite) region. We map it conformally onto the interior D (or exterior D') of a unit circle with circumference γ using the functions

$$z = w(\zeta), \quad w'(\zeta) \neq 0, \quad \zeta = r\exp(i\theta) \in D. \tag{3.17}$$

Then the complex potentials take the form

$$\varphi(z) = \varphi(\zeta), \ \Phi(z) = \varphi'(z) = \varphi'(\zeta)/w'(\zeta) = \Phi(\zeta), \ \Phi'(z) = \varphi''(z) = \Phi'(\zeta)/w'(\zeta),$$
$$\psi(z) = \psi(\zeta), \quad \Psi(z) = \psi'(z) = \psi'(\zeta)/w'(\zeta) = \Psi(\zeta),$$

while the displacements of Eq. (2.7), stresses of Eq. (2.10), rotations and elongation-shears of Eq. (2.11) (in the absence of volume forces) will be, respectively,

$$\begin{gathered}
2\mu u = \varkappa\varphi(\zeta) - w(\zeta)\overline{\Phi(\zeta)} - \overline{\psi(\zeta)} - \alpha\,[w(\zeta)\bar\Phi^2(\bar\zeta) - \\
-2\varphi(\zeta)\overline{\Phi(\zeta)} + \textstyle\int \Phi(\zeta)\varphi'(\zeta)\,d\zeta\,], \\
P^{11} = \overline{P^{22}} = -\frac{2}{\overline{w'(\zeta)}}\left\{w(\zeta)\overline{\Phi'(\zeta)} + \overline{\psi'(\zeta)} + 2\alpha\overline{\Phi'(\zeta)}\,[w(\zeta)\overline{\Phi(\zeta)} - \varphi(\zeta)]\right\}, \\
P^{12} = 2\,[\Phi(\zeta) + \overline{\Phi(\zeta)}] + 2\alpha\,[\Phi(\zeta) - \overline{\Phi(\zeta)}]^2, \\
\mu\omega^{21} = 2(1-\nu)\,[\Phi(\zeta) - \overline{\Phi(\zeta)}], \\
2\mu e^{11} = 2\mu\overline{e^{22}} = P^{11}, \quad 2\mu e^{12} = (1-2\nu)P^{12} - \mu(\omega^{21})^2/2.
\end{gathered} \tag{3.18}$$

In view of Eq. (3.17) the polar coordinates r, θ of the plane ζ are orthogonal to the curvilinear coordinates in the plane z. The physical components of the displacements and stresses in these coordinates are related to the complex potentials by the expressions [4]

$$u_r + iu_\theta = \frac{\bar\zeta}{|\zeta|}\frac{\overline{w'(\zeta)}}{|w'(\zeta)|}u, \tag{3.19}$$

$$P_{rr} - P_{\theta\theta} + 2iP_{r\theta} = \frac{\zeta}{\bar\zeta}\frac{\overline{w'(\zeta)}}{w'(\zeta)}P^{11}, \quad P_{rr} + P_{\theta\theta} = P^{12}.$$

For the mapping of Eq. (3.17) the boundary problems for the potentials for specification of boundary displacements (2.12) or boundary stresses (3.8) take on the form of problems on the circumference of a unit circle:

$$\begin{aligned} &\varkappa\varphi(\tau) - w(\tau)\overline{\varphi'(\tau)}/\overline{w'(\tau)} - \overline{\psi(\tau)} - \alpha N(\varphi(\tau), \overline{\varphi(\tau)}) = h_0(\tau); \\ &\varphi(\tau) + w(\tau)\overline{\varphi'(\tau)}/\overline{w'(\tau)} + \overline{\psi(\tau)} + \alpha N(\varphi(\tau), \overline{\varphi(\tau)}) = g_0(\tau). \end{aligned} \quad (3.20)$$

Here

$$N(\varphi(\tau), \overline{\varphi(\tau)}) = w(\tau)\frac{\bar\varphi'^2(\bar\tau)}{\bar w'^2(\bar\tau)} - 2\varphi(\tau)\frac{\overline{\varphi'(\tau)}}{\overline{w'(\tau)}} + \int \varphi'^2(\tau)\frac{d\tau}{w'(\tau)}; \quad (3.21)$$
$$\tau = \exp(i\theta) \in \gamma;$$

$h_0(\tau)$ and $g_0(\tau)$ are functions known on γ, the constant appearing in the latter being arbitrarily fixed.

4. We will consider the problem of volume extension of an elastic plane with an orifice which has the shape of an ellipse in the deformed state in the absence of stresses on the orifice contour, rotation at infinity, and volume forces.

Let a and b ($a > b$) be the semiaxes of the ellipse. We use a Cartesian coordinate system coinciding with those axes, directing the abscissa along the major axis. Then the equations of the ellipse (contour L) and the parameters n, m, characterizing its dimensions and form, have the appearance

$$\begin{aligned} &x^2/a^2 + y^2/b^2 = 1, \quad 0 < n = (a+b)/2 < \infty, \\ &0 < m = (a-b)/(a+b) < 1. \end{aligned} \quad (4.1)$$

In the limiting cases, at m = 0 the ellipse degenerates into a circle, and at m = 1, into a rectilinear slot.

We denote by $P_0 > 0$ the tensile stresses and consider that in accordance with the initial data V = 0, W = 0, $p_x = p_y = 0$. Then the conditions on the periphery of the orifice (3.8) (for zero constant) and at infinity take on the form

$$g_x^0 = 0, \quad g_y^0 = 0; \quad (4.2)$$

$$P_{xx}^\infty = P_{yy}^\infty = P_0, \quad P_{xy}^\infty = 0, \quad \omega_{xy}^\infty = 0. \quad (4.3)$$

The exterior of the ellipse is reflected conformally onto the exterior of a unit circle by means of the function

$$z = w(\zeta) = n(\zeta + m/\zeta), \quad w'(\zeta) \neq 0, \quad \zeta = r\exp(i\theta) \in D' \quad (4.4)$$

whence follow the expressions

$$x = n(r + m/r)\cos\theta, \quad y = n(r - m/r)\sin\theta; \quad (4.5)$$

$$\begin{aligned} &x^2/(n(r + m/r))^2 + y^2/(n(r - m/r))^2 = 1, \\ &x^2/(2n\sqrt{m}\cos\theta)^2 - y^2/(2n\sqrt{m}\sin\theta)^2 = 1, \end{aligned} \quad (4.6)$$

indicating that r, θ are elliptical coordinates in the plane of deformation and that the boundary ellipse of Eq. (4.1) corresponds to the ellipse r = 1.

With consideration of conditions (4.2), which in complex form appear as $g_0 = 0$, the second boundary condition for the potentials of Eq. (3.20) becomes homogeneous:

$$\begin{aligned} &\varphi(\zeta) + w(\zeta)\overline{\Phi(\zeta)} + \overline{\psi(\zeta)} + \alpha\,[w(\zeta)\bar\Phi^2(\bar\zeta) - 2\varphi(\zeta)\overline{\Phi(\zeta)} + \\ &+ \int \Phi(\zeta)\varphi'(\zeta)\,d\zeta\,]\Big|_{r=1} = 0. \end{aligned} \quad (4.7)$$

We take the potential $\varphi(\zeta)$ in a form which generalizes its expression in the linear solution of this problem [4] and contains two free real parameters A and B, so that Eq. (4.7) defines the other potential $\psi(\zeta)$, and finally

$$\varphi(\zeta) = n\left(A\zeta + \frac{mB}{\zeta}\right), \quad \psi(\zeta) = -n\left[A\frac{1+\alpha A}{\zeta} + \right.$$
$$+ mB(1+\alpha B)\zeta + \left(\frac{1-2\alpha A}{\zeta} + m(1-2\alpha B)\zeta\right)\frac{A\zeta^2 - mB}{\zeta^2 - m} + \tag{4.8}$$
$$\left. + \alpha\frac{1+m\zeta^2}{\zeta}\left(\frac{A\zeta^2 - mB}{\zeta^2 - m}\right)^2 + \frac{\alpha\sqrt{m}}{2}(A-B)^2 \ln\frac{1-\sqrt{m}\zeta}{1+\sqrt{m}\zeta}\right].$$

In the absence of volume forces the potentials of Eq. (4.8) and the mapping function (4.4) correspond to the following stress expressions of Eq. (3.18):

$$p^{12} = 2\left(\frac{A\zeta^2 - mB}{\zeta^2 - m} + \frac{A\bar\zeta^2 - mB}{\bar\zeta^2 - m}\right) + 2\alpha\left(\frac{A\zeta^2 - mB}{\zeta^2 - m} - \frac{A\bar\zeta^2 - mB}{\bar\zeta^2 - m}\right)^2,$$
$$p^{11} = \frac{2\bar\zeta^2}{\bar\zeta^2 - m}\left\{\frac{2m(A-B)(\zeta\bar\zeta - 1)}{(\bar\zeta^2 - m)^2}\left[\frac{\zeta - m\bar\zeta}{\zeta} + 2\alpha\frac{m(A-B)(\zeta - \bar\zeta^3)}{\zeta(\bar\zeta^2 - m)}\right] + \right.$$
$$+ \frac{m(A+B)(\bar\zeta^4 + 1) - 2(A + m^2B)\bar\zeta^2}{\bar\zeta^2(\bar\zeta^2 - m)} + \alpha\left[\frac{(A - mB\bar\zeta^2)^2}{\bar\zeta^2(m\bar\zeta^2 - 1)} + \right. \tag{4.9}$$
$$\left.\left. + 2\frac{A - mB\bar\zeta^2}{\bar\zeta^2}\frac{A\zeta^2 - mB}{\bar\zeta^2 - m} + \frac{m\bar\zeta^2 - 1}{\bar\zeta^2}\left(\frac{A\bar\zeta^2 - mB}{\bar\zeta^2 - m}\right)^2\right]\right\}.$$

We arrange the arbitrariness of the parameters A and B to satisfy the conditions at infinity for the stresses of Eq. (4.3), which in complex form appear as $P_\infty{}^{11} = 0$, $P_\infty{}^{12} = 2P_0$. As a result we obtain the equations

$$\alpha(A-B)^2 + A + B = 0, \quad 2A = P_0,$$

defining these parameters:

$$A = \frac{P_0}{2}, \quad B^\pm = P_0\left(\frac{1}{2} + \frac{-1 \pm \sqrt{1 - 2\sigma(1-\nu)}}{\sigma(1-\nu)}\right), \quad \sigma = \frac{P_0}{\mu}. \tag{4.10}$$

Equations (4.9), (4.10) yield two solutions for the problem. The solutions found remain in force for limited extensions, for which $\sigma \leq 1/(2(1-\nu))$ (at $\sigma = 1/(2(1-\nu))$ they coincide with each other), and lose meaning at intense extensions for which $\sigma > 1/(2(1-\nu))$ and the parameters of Eq. (4.10) become complex. It will be shown below that the second solution does not correspond to the original assumptions of the model considered and should be rejected.

The potentials of Eq. (4.8) in the absence of forces correspond to the following rotations and elongation-shears of Eq. (3.18):

$$\omega^{21} = 2\frac{1-\nu}{\mu}\frac{m(A-B)(\bar\zeta^2 - \zeta^2)}{(\zeta^2 - m)(\bar\zeta^2 - m)},$$
$$e^{12} = \frac{1-2\nu}{\mu}\frac{2(A\zeta^2\bar\zeta^2 + m^2B) - m(A+B)(\zeta^2 + \bar\zeta^2)}{(\zeta^2 - m)(\bar\zeta^2 - m)} - \frac{1-\nu}{2\mu^2}\frac{m(A-B)^2(\bar\zeta^2 - \zeta^2)^2}{(\zeta^2 - m)^2(\bar\zeta^2 - m)^2},$$
$$e^{11} = \frac{\bar\zeta^2}{\bar\zeta^2 - m}\left\{\frac{2m(A-B)(\zeta\bar\zeta - 1)}{\mu(\bar\zeta^2 - m)^2}\left[\frac{\zeta - m\bar\zeta}{\zeta} + \frac{1-\nu}{\mu}\frac{m(A-B)(\zeta - \bar\zeta^3)}{\zeta(\bar\zeta^2 - m)}\right] + \right. \tag{4.11}$$
$$+ \frac{m(A+B)(\bar\zeta^4 + 1) - 2(A + m^2B)\bar\zeta^2}{\mu\bar\zeta^2(\bar\zeta^2 - m)} + \frac{1-\nu}{2\mu^2}\left[\frac{(A - mB\bar\zeta^2)^2}{\bar\zeta^2(m\bar\zeta^2 - 1)} + \right.$$

$$+ 2\frac{A - mB\bar{\zeta}^2}{\bar{\zeta}^2}\frac{A\bar{\zeta}^2 - mB}{\bar{\zeta}^2 - m} + \frac{m\bar{\zeta}^2 - 1}{\bar{\zeta}^2}\left(\frac{A\bar{\zeta}^2 - mB}{\bar{\zeta}^2 - m}\right)^2\Bigg]\Bigg\}.$$

Here the parameters A and B are defined by Eq. (4.10). From this it is evident that ${\omega_\infty}^{21} = 0$, i.e., the condition for rotation at infinity, Eq. (4.3), is not satisfied.

It is also evident from Eq. (4.11) that to realize the first initial assumption of the model $|e^{\alpha\beta}| \ll 1$, $|\omega^{\alpha\beta}| \ll 1$, we must take

$$|A|/\mu \ll 1, \quad |B|/\mu \ll 1. \tag{4.12}$$

It follows from the first condition of Eq. (4.12) with consideration of Eq. (4.10) that the characteristic dimensionless stress must be small in comparison to the unit quantity

$$|A|/\mu = \sigma/2 \ll 1, \quad \sigma = P_0/\mu \ll 1. \tag{4.13}$$

Note that the Poisson coefficient ν and the quantity $1/(2(1 - \nu))$ for elastic materials vary over the ranges

$$0 < \nu < 0{,}5, \quad 0{,}5 < 1/(2\,(1 - \nu)) < 1,$$

so that the case of coinciding solutions in which $\sigma = 1/(2(1 - \nu))$ does not agree with Eq. (4.13) and must be rejected. In the future we will consider only weak extension where according to Eq. (4.13), $\sigma \ll 1$. In this case the expression $B^{\pm}/\mu$ can be linearized with respect to σ and written in the form

$$\frac{B^+}{\mu} = \frac{\sigma}{2} + \frac{1}{1 - \nu}\left(-1 + \sqrt{1 - 2\sigma\,(1 - \nu)}\right) \approx -\frac{1}{2}\sigma,$$
$$\frac{B^-}{\mu} = \frac{\sigma}{2} - \frac{1}{1 - \nu}\left(1 + \sqrt{1 - 2\sigma\,(1 - \nu)}\right) \approx -\frac{2}{1 - \nu} + \frac{3}{2}\sigma. \tag{4.14}$$

It can easily be seen from Eqs. (4.14), (4.13) that the second condition of Eq. (4.12) is satisfied in the first solution but not in the second. Thus, both requirements of Eq. (4.12) for slight extension correspond only to the first solution, in which we must take

$$\sigma \ll 1, \quad A = P_0/2, \quad B = -P_0/2. \tag{4.15}$$

For the parameter values of Eq. (4.15) for the rotation and elongation-shears of Eq. (4.11) in the elliptical coordinates of Eqs. (4.5), (4.6) we obtain the expressions

$$\omega^{21} = -i\sigma\,\frac{4m\,(1 - \nu)\,r^2 \sin 2\theta}{r^4 - 2mr^2 \cos 2\theta + m^2},$$
$$e^{12} = 2\sigma\,\frac{(1 - 2\nu)\,(r^4 - m^2) + \sigma\,(1 - \nu)\,m^2 r^4 \sin^2 2\theta}{r^4 - 2mr^2 \cos 2\theta + m^2},$$
$$e^{11} = \frac{\sigma r^2 e^{-2i\theta}}{r^2 e^{-2i\theta} - m}\left\{\frac{2m\,(r^2 - 1)}{(r^2 e^{-2i\theta} - m)^2}\left[1 - me^{-2i\theta} + \sigma\,\frac{m\,(1 - \nu)\,(1 - r^2 e^{-2i\theta})}{r^2 e^{-2i\theta} - m}\right] - \frac{1 - m^2}{r^2 e^{-2i\theta} - m} + \right.$$
$$\left. + \sigma\,\frac{1 - \nu}{8}\left[\frac{(1 + mr^2 e^{-2i\theta})^2}{r^2 e^{-2i\theta}\,(mr^2 e^{-2i\theta} - 1)} + \frac{r^2 e^{-2i\theta} - m}{r^2 e^{-2i\theta}}\,\frac{3m\,(r^4 e^{-4i\theta} - 1) + (1 - m)\,r^2 e^{-2i\theta}}{(r^2 e^{-2i\theta} - m)^2}\right]\right\}. \tag{4.16}$$

From Eq. (4.16) with consideration of the relationship

$$r^4 - m^2 = (r^2 + m)\,[(r + 1)\,(r - 1) + (1 - m)]$$

we can conclude that under the conditions

$$r - 1 \sim \sigma, \quad 1 - m \sim \sigma, \quad 2\theta \neq 0;\ \pi;\ 2\pi;\ 3\pi$$

we have $|e^{\alpha\beta}| \sim |\omega^{21}|^2$, i.e., the other original assumption of the model will be satisfied. Thus, for slight extension of a plane with prolate elliptical orifice there exists a region in the form of a band enclosing the ellipse periphery except for the endpoints of the axes of symmetry, within which small rotations significantly exceed small elongation-shears. Thus, solution of the problem within the framework of the nonlinear model considered is justified.

The physical components of the stresses in the elliptical coordinates P_{rr}, $P_{r\theta}$, $P_{\theta\theta}$ are related to the complex components by Eqs. (3.19). If we substitute in them the reflection of Eq. (4.4), the complex stresses of Eq. (4.9), and parameters of Eq. (4.15), then the solution of the problem corresponding to the original assumptions of the nonlinear model will be defined by the expressions

$$\begin{aligned}
P_{rr} + P_{\theta\theta} &= \frac{2P_0 (r^4 - m^2)}{(r^2 e^{2i\theta} - m)(r^2 e^{-2i\theta} - m)} + \sigma \frac{P_0 (1 - \nu) m^2 r^4 (e^{2i\theta} - e^{-2i\theta})^2}{(r^2 e^{2i\theta} - m)^2 (r^2 e^{-2i\theta} - m)^2}, \\
P_{rr} - P_{\theta\theta} + 2iP_{r\theta} &= \frac{2P_0 r^2}{r^2 e^{2i\theta} - m} \left\{ \frac{2m (r^2 - 1)}{(r^2 e^{-2i\theta} - m)^2} \left[1 - m e^{-2i\theta} + \sigma m (1 - \nu) \frac{1 - r^2 e^{-4i\theta}}{r^2 e^{-2i\theta} - m} \right] - \right. \\
&\quad - \frac{1 - m^2}{r^2 e^{-2i\theta} - m} - \sigma e^{2i\theta} \frac{1 - \nu}{8r^2} \left[\frac{(1 + m r^2 e^{-2i\theta})^2}{1 - m r^2 e^{-2i\theta}} - 2 (1 + m r^2 e^{-2i\theta}) \frac{r^2 e^{-2i\theta} + m}{r^2 e^{-2i\theta} - m} + \right. \\
&\quad \left. \left. + (1 - m r^2 e^{-2i\theta}) \left(\frac{r^2 e^{-2i\theta} + m}{r^2 e^{-2i\theta} - m} \right)^2 \right] \right\}.
\end{aligned} \tag{4.17}$$

For the stresses $P_{rr}{}^1$, $P_{r\theta}{}^1$, $P_{\theta\theta}{}^1$ on the boundary ellipse $r = 1$ Eq. (4.17) yields the expressions

$$P^1_{rr} = P^1_{r\theta} = 0, \quad P^1_{\theta\theta} = \frac{2P_0 (1 - m^2)}{1 - 2m \cos 2\theta + m^2} - \sigma \frac{4P_0 m (1 - \nu) \sin^2 2\theta}{(1 - 2m \cos 2\theta + m^2)^2}. \tag{4.18}$$

We will study the behavior of the stress $P_{\theta\theta}{}^1$ on the boundary of the orifice. The extremal points are defined by the equation $dP_{\theta\theta}{}^1/d2\theta = 0$, which is satisfied for one of the two conditions

$$\sin 2\theta = 0, \quad \cos 2\theta = v_1/v_2,$$

where

$$v_1 = 1 - m^4 - 4m^2\sigma (1 - \nu); \quad v_2 = 2m\,[1 - m^2 - \sigma (1 - \nu)(1 + m^2)].$$

In the interval $0 \leq 2\theta \leq 4\pi$ these equations have the roots

$$2\theta = 0;\ \pi;\ 2\pi;\ 3\pi,$$
$$2\theta = \pm\arccos v_1/v_2; \quad 2\pi \pm \arccos v_1/v_2,$$

of which the following are realized for $|v_2| > |v_1|$, i.e., for

$$v_2 > 0, \quad -v_2 < v_1 < v_2$$

or for

$$v_2 < 0, \quad v_2 < v_1 < -v_2.$$

These systems of inequalities, having been written with consideration of the expressions for v_1, v_2, when solved for the quantity $\sigma(1 - \nu)$, agree with each other for

$$\sigma (1 - \nu) < -\frac{1 - m^2}{2m}, \quad \sigma (1 - \nu) > \frac{1 - m^2}{2m}. \tag{4.19}$$

respectively. Since $\sigma(1 - \nu) > 0$, only the second of these is meaningful. Solving the latter for m with consideration of the conditions $m > 0$ and $\sigma \ll 1$, in the approximation linear in σ we obtain $m > 1 - \sigma(1 - \nu)$. Standard analysis reveals that the nonzero stress of Eq. (4.18) for one form of orifice

$$0 < m \leqslant 1 - \sigma(1 - \nu)$$

has extrema

$$\begin{aligned} &\text{at} \quad 2\theta = 0;\ 2\pi \quad P^1_{\theta\theta} = (P^1_{\theta\theta})_{\max} = 2P_0 \frac{1+m}{1-m} > 0, \\ &\text{at} \quad 2\theta = \pi;\ 3\pi \quad P^1_{\theta\theta} = (P^1_{\theta\theta})_{\min} = 2P_0 \frac{1-m}{1+m} > 0, \end{aligned} \tag{4.20}$$

i.e., prolate elliptical orifices extend weakly and moderately. As in the linear solution of [4] the extensions are extremal at the points of the ellipse lying on its axes of symmetry, with the maximum on the major semiaxis and the minimum on the minor. In particular, it follows from Eq. (4.20) that upon degeneration of the ellipse into a circle ($m = 0$) the extrema coincide in magnitude and are equal to double the stress at infinity.

For another orifice form

$$1 - \sigma(1 - \nu) < m < 1$$

we obtain

$$\begin{aligned} &\text{at } 2\theta = 0;\ 2\pi \quad P^1_{\theta\theta} = (P^1_{\theta\theta})_{\max} = 2P_0 \frac{1+m}{1-m} > 0, \\ &\text{at } 2\theta = \pi;\ 3\pi \quad P^1_{\theta\theta} = (P^1_{\theta\theta})_{\max} = 2P_0 \frac{1-m}{1+m} > 0, \\ &\text{at } 2\theta = \pm \arccos \frac{v_1}{v_2};\quad 2\pi \pm \arccos \frac{v_1}{v_2} \\ &P^1_{\theta\theta} = (P^1_{\theta\theta})_{\min} = -\frac{P_0}{\sigma(1-\nu)} (\tau - 1/m)(\tau - m), \end{aligned} \tag{4.21}$$

where in view of the second inequality of Eq. (4.19)

$$\tau = 2m\sigma(1 - \nu)/(1 - m^2) > 1, \ \tau - m > 0.$$

Hence, it is clear that

$$\begin{aligned} &\text{at} \quad \tau < 1/m \quad (P^1_{\theta\theta})_{\min} > 0, \\ &\text{at} \quad \tau > 1/m \quad (P^1_{\theta\theta})_{\min} < 0, \\ &\text{at} \quad \tau = 1/m, \quad \tau = m \quad (P^1_{\theta\theta})_{\min} = 0. \end{aligned}$$

Thus, on the contours of highly prolate elliptical orifices the properties of the limiting stress are more diverse. In this case stress maxima occur at the ends of the axes of symmetry, with absolute maxima occurring on the major axis. In the vicinity of these points the contour is in tension.

Stress minima identical in magnitude are reached at points occupying intermediate positions between the maximum points. In the vicinities of these points, depending on the parameter values, the contour may be either in tension, a neutral state, or compression. In particular, it is evident from Eq. (4.21) that upon degeneration of the ellipse into a rectilinear slot ($m = 1$) the absolute stress maximum increases without limit, while the minimum decreases without limit, these extrema being realized at one and the same points, the ends of the slot. Thus, the stress becomes undefined at the slot ends. The stress in the middle of the slot will equal zero.

5. We will assume that volume forces are absent, that an elastic plane has an orifice of arbitrary form, and consider the boundary problem for the potentials of Eq. (3.8) for stresses finite at infinity and having values of the general form at the orifice boundary.

Generally speaking, in a singly-connected finite region (not containing the origin of the reference system) the complex potentials are not unique. However, the requirements of unambiguous stress and rotation values impose certain limitations on them.

We denote by $[\chi]_l$ the increment in the smooth function $\chi(z, \bar{z})$ upon positive traversal of the contour l containing the orifice. It is known [6] that for a smooth function the relative order of the operation of calculating the increment along a closed contour can be transposed with that of differentiation:

$$\frac{\partial}{\partial z}[\chi]_l = \left[\frac{\partial \chi}{\partial z}\right]_l, \quad \frac{\partial}{\partial \bar{z}}[\chi]_l = \left[\frac{\partial \chi}{\partial \bar{z}}\right]_l. \tag{5.1}$$

When volume forces are absent (V = 0) the conditions for nonambiguity of the stresses and rotation of Eqs. (3.4) and (3.7) have the form

$$0 = \left[P^{12} + \frac{1}{2\alpha}\omega^{21}\left(1 - \frac{1}{4}\omega^{21}\right)\right]_l = 4\,[\varphi'(z)]_l; \tag{5.2}$$

$$0 = [P^{11}]_l = -2\,[z\overline{\varphi''(z)} + \overline{\psi'(z)} + 2\alpha\overline{\varphi''(z)}\,(z\overline{\varphi'(z)} - \varphi(z))]_l. \tag{5.3}$$

From Eqs. (5.2) and (5.1) it follows that

$$[\varphi'(z)]_l = 0, \quad [\varphi''(z)]_l = 0. \tag{5.4}$$

Then a consequence of Eqs. (5.3), (5.4) will be the relationship

$$[\overline{\psi'(z)}]_l - 2\alpha\overline{\varphi''(z)}\,[\varphi(z)]_l = 0,$$

which defines the function increments in the form

$$[\varphi(z)]_l = 0, \quad [\psi'(z)]_l = 0. \tag{5.5}$$

With consideration of Eqs. (5.4), (5.5) the complex potentials can be represented in terms of functions well-defined in an infinite region φ^* and ψ^*:

$$\varphi(z) = \varphi^*(z), \quad \psi(z) = B\ln z + \psi^*(z), \quad B = \text{const}. \tag{5.6}$$

Expanding these functions in Loran series, we find that finiteness of the stresses and rotation of Eqs. (3.4), (3.7) in the infinite region requires that the potentials of Eq. (5.6) have the form

$$\begin{gathered}\varphi(z) = a_1 z + \varphi_*(z), \quad \psi(z) = B\ln z + b_1 z + \psi_*(z),\\ \varphi_*(z) = \sum_0^\infty a_{-n}z^{-n}, \quad \psi_*(z) = \sum_0^\infty b_{-n}z^{-n},\end{gathered} \tag{5.7}$$

where the coefficients a_1 and b_1 are defined by the conditions at infinity as follows:

$$a_1 = \frac{1}{4}\left[P^{12}_\infty + \frac{1}{2\alpha}\omega^{21}_\infty\left(1 - \frac{1}{4}\omega^{21}_\infty\right)\right], \quad b_1 = -\frac{1}{2}P^{22}_\infty. \tag{5.8}$$

The coefficient B can also be expressed in terms of a mechanical quantity. For this purpose we will consider the arc $\overset{\smile}{A_0A}$ located in the region S and assume that at each point thereon the stress vector is defined. Then in accord with Eq. (3.3) the components of the main vector and main moment of the forces distributed along the arc are defined by the expressions

$$F = F_x + iF_y = \int_0^s (p_x + ip_y)\, ds = \int_0^s p ds = \frac{1}{2i}\int_0^s \left(P^{12}\frac{dz}{ds} - P^{11}\frac{d\bar{z}}{ds}\right) ds,$$

$$M = M_z = \int_0^s \left(xp_y - yp_x\right) ds = \mathrm{Re}\left\{-i\int_0^s \bar{z}p ds\right\} =$$

$$= \mathrm{Re}\left\{-\frac{1}{2}\int_0^s \bar{z}\, \bar{z}\left(P^{12}\frac{dz}{ds} - P^{11}\frac{d\bar{z}}{ds}\right) ds\right\}.$$

Using the stress expressions of Eqs. (3.5) and (3.7) in the absence of volume forces we find the integrands

$$\frac{1}{2}\left(P^{12}\frac{dz}{ds} - P^{11}\frac{d\bar{z}}{ds}\right) = \frac{d}{ds}\{\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} +$$
$$+ \alpha\,[z\bar{\varphi}'^2(\bar{z}) - 2\varphi(z)\overline{\varphi'(z)} + \int \varphi'^2(z)\, dz]\},$$
$$\mathrm{Re}\left\{\frac{\bar{z}}{2}\left(P^{12}\frac{dz}{ds} - P^{11}\frac{d\bar{z}}{ds}\right)\right\} = \mathrm{Re}\frac{d}{ds}\{z\bar{z}\varphi'(z) + z\psi(z) -$$
$$-\int \psi(z)\, dz + \alpha\,(z\bar{z}\,\varphi'^2(z) - 2z\overline{\varphi(z)}\,\varphi'(z) + \varphi(z)\,\overline{\varphi(z)})\},$$

and consequently, the components of the main vector and main moment of contour forces are defined by the potentials with the expressions

$$\begin{aligned} iF &= \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + \alpha\,[z\bar{\varphi}'^2(\bar{z}) - 2\varphi(z)\,\overline{\varphi'(z)} + \\ &+ \int \varphi'^2(z)\, dz]\,|_{z(s)} + \text{const}, \\ M &= -\mathrm{Re}\,\{z\bar{z}\varphi'(z) + z\psi(z) - \int \psi(z)\, dz + \\ &+ \alpha\,(z\bar{z}\varphi'^2(z) - 2z\overline{\varphi(z)}\,\varphi'(z) + \varphi(z)\overline{\varphi(z)})\}\,|_{z(s)} + \text{const}. \end{aligned} \tag{5.9}$$

Applying the first expression of Eq. (5.9) to the closed contour L, traversed in the positive direction (i.e., clockwise), and using the expressions which follow from Eq. (5.7)

$$[\psi(z)]_L = 2\pi iB, \quad \left[\int \varphi'^2(z)\, dz\right]_L = 0,$$

we find that the constant B can be expressed as

$$B = \bar{F}/2\pi. \tag{5.10}$$

The potentials of Eq. (5.7) correspond to the following displacements of Eq. (3.9):

$$2\mu u = R\,((\varkappa a_1 - \bar{a}_1)\, e^{i\sigma} - \bar{b}_1 e^{-i\sigma}) - B\ln R + \varkappa a_0 + i\bar{B}\sigma + O(1/R), \quad z = Re^{i\sigma}.$$

Here by O(1/R) we denote terms which are of order 1/R outside L. Hence it is evident that, generally speaking, for finite stresses and rotations the displacements increase without limit at infinity. In order that they remain finite, the conditions

$$a_1 = 0, \quad b_1 = 0, \quad B = 0. \tag{5.11}$$

must be observed. In light of Eq. (5.8) the first two of these require that stresses and rotation disappear at infinity, while in view of Eq. (5.10) the last expression requires that the main vector of the contour forces vanish. In the case of Eq. (5.11) the potentials of Eq. (5.7) are holomorphic in the infinite region of the functions, and at infinity the displacement takes on a value defined by the coefficient a_0:

$$\varphi(z) = \varphi_*(z) = \sum_0^\infty a_{-n} z^{-n}, \quad \psi(z) = \psi_*(z) = \sum_0^\infty b_{-n} z^{-n}, \quad 2\mu u_\infty = \varkappa a_0. \tag{5.12}$$

Thus, for stresses, rotations, and displacements finite within an infinite region the potentials are well-defined, have the form of Eq. (5.12), and are defined in terms of boundary condition (3.8), where we must take V = 0, W = 0, const = 0.

If for finite stresses and rotations infinite displacements at infinity are admissible, then the potentials are unambiguous and defined by Eq. (5.7). In this case we can consider the boundary problem for the unique potentials φ_* and ψ_*, which follows from the problem of Eq. (3.8) and is of the same type:

$$A_*\varphi_*(z) + \overline{A}_* z\overline{\varphi'(z)} + \overline{\psi_*(z)} + \alpha N(\varphi_*, \overline{\varphi}_*)|_L = g_*(s).$$

Here

$$A_* = 1 + 2\alpha(a_1 - \overline{a}_1); \quad N(\varphi_*, \overline{\varphi}_*) = z\overline{\varphi}_*'^2(\overline{z}) - 2\varphi_*(z)\overline{\varphi_*'(z)} + \int \varphi_*'^2(z)\, dz;$$
$$g_*(s) = g(s) - [a_1 + \overline{a}_1 + \alpha(a_1 - \overline{a}_1)^2] z(s) - \overline{B} \ln \overline{z}(s) - \overline{b}_1 \overline{z}(s);$$

g_* is a known function well-defined on L; therefore in the future we will limit our examination to the problem of Eq. (3.8).

The infinite singly-connected region can be conformally mapped onto a unit circle D (with circumference γ) by the function [4]

$$z = w(\zeta) = \frac{c}{\zeta} + w_0(\zeta), \quad \zeta \in D,$$

where c is a constant and w_0 is a function holomorphic on the circle. Then the potentials of Eq. (5.7) are defined by the equalities

$$\varphi(\zeta) = \frac{a_1 c}{\zeta} + \varphi_0(\zeta), \quad \psi(\zeta) = -B \ln \zeta + \frac{b_1 c}{\zeta} + \psi_0(\zeta)$$

[$\varphi_0(\zeta)$, $\psi_0(\zeta)$ are functions holomorphic within the circle], the stresses and rotation are defined by Eq. (3.18), and given Eq. (5.11), the boundary condition for the potentials is defined by the second expression of Eq. (3.20).

We will use condition (3.20) to seek the potentials $\varphi(\zeta)$ and $\psi(\zeta)$, which together with their derivatives are continuous in the closed region and satisfy the condition $\psi(0) = 0$ (without affecting the form of the stresses and rotations). We multiply condition (3.20) itself and the equation complex conjugate thereto by $1/(2\pi i(\tau - \zeta))$ and integrating each along the contour γ by using the properties of a holomorphic function

$$\frac{1}{2\pi i} \int_\gamma \frac{v(\tau)\, d\tau}{\tau - \zeta} = v(\zeta), \quad \frac{1}{2\pi i} \int_\gamma \frac{\overline{v(\tau)}\, d\tau}{\tau - \zeta} = \overline{v(0)},$$

we obtain expressions for the potentials in terms of the boundary values of one of them and its derivative:

$$\begin{aligned} \varphi(\zeta) &= \frac{1}{2\pi i} \int_\gamma \frac{g(\tau)\, d\tau}{\tau - \zeta} - \frac{1}{2\pi i} \int_\gamma \frac{\overline{\varphi'(\tau)}}{\overline{w'(\tau)}} \frac{w(\tau)\, d\tau}{\tau - \zeta} - \frac{\alpha}{2\pi i} \int_\gamma \frac{N(\tau)\, d\tau}{\tau - \zeta}; \\ \psi(\zeta) &= \frac{1}{2\pi i} \int_\gamma \frac{\overline{g(\tau)}\, d\tau}{\tau - \zeta} - \frac{1}{2\pi i} \int_\gamma \frac{\varphi'(\tau)}{w'(\tau)} \frac{\overline{w(\tau)}\, d\tau}{\tau - \zeta} - \frac{\alpha}{2\pi i} \int_\gamma \frac{\overline{N(\tau)}\, d\tau}{\tau - \zeta} - \overline{\varphi(0)} \end{aligned} \tag{5.13}$$

[$N(\tau)$ is defined by Eq. (3.21)]. The first relationship of Eq. (5.13), written in the form

$$P(\varphi(\zeta), \alpha) = \Pi(\varphi(\zeta)) + \alpha R(\varphi(\zeta)) = 0,$$

where

$$\Pi(\varphi(\zeta)) = \varphi(\zeta) + \frac{1}{2\pi i}\int_\gamma K(\zeta, \tau)\,\overline{\varphi'(\tau)}\,d\tau - A(\zeta),$$

$$R(\varphi(\zeta)) = \frac{1}{2\pi i}\int_\gamma K(\zeta, \tau)\frac{\overline{\varphi'}^2(\tau)}{\overline{w'(\tau)}}\,d\tau - \frac{1}{\pi i}\int_\gamma \frac{\overline{\varphi'(\tau)}}{\overline{w'(\tau)}}\frac{\varphi(\tau) - \varphi(\zeta)}{\tau - \zeta}\,d\tau + \int \frac{\varphi'^2(\zeta)}{w'(\zeta)}\,d\zeta,$$

$$K(\zeta, \tau) = \frac{1}{\overline{w'(\tau)}}\frac{w_0(\tau) - w_0(\zeta)}{\tau - \zeta}, \quad A(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{g(\tau)\,d\tau}{\tau - \zeta},$$

after direction of ζ to a point σ on the boundary circle, leads to a nonlinear functional equation for the unknown function $\varphi(\sigma)$ which contains the parameter α as a factor before the nonlinear terms:

$$P(\varphi(\sigma), \alpha) = \Pi(\varphi(\sigma)) + \alpha R(\varphi(\sigma)) = 0 \tag{5.14}$$

[$\Pi(\varphi)$ is the linear elasticity operator].

We will apply the modified Newton's method of [7] to Eq. (5.14), seeking the solution in the form of a series

$$\varphi_{n+1} = \varphi_n - [\Pi'(\varphi_0)]^{-1}(P(\varphi_n)) \quad (n = 0, 1, \ldots), \tag{5.15}$$

in which for the zeroeth approximation we take the solution φ_0 of the linear problem $\Pi(\varphi_0) = 0$. The process of Eq. (5.15) has the advantage that it makes use of an inverse operator corresponding to the zeroeth approximation. For convergence of the sequence of Eq. (5.15) it is required that the parameter not exceed some number defined by the form of the operators contained in the equations.

REFERENCES

1. V. V. Novozhilov, Theory of Elasticity [in Russian], Sulpromiz, Leningrad (1958).
2. I. G. Petrovskii, Lectures on Equations with Partial Derivatives [in Russian], GIFML, Moscow (1961).
3. V. D. Bondar', "The planar problem of one variant of geometrically nonlinear elasticity," in: Dynamics of Continuous Media: Collected Studies [in Russian], Sib. Otdel., Akad. Nauk SSSR, Inst. Gidrodinamiki (1969).
4. N. I. Muskhelishvili, Some Fundamental Problems in the Theory of Elasticity [in Russian], Nauka, Moscow (1966).
5. G. V. Kolosov, On an Application of the Theory of Complex Variables to the Planar Problem of the Mathematical Theory of Elasticity [in Russian], Yur'ev (1909).
6. I. N. Sneddon and P. S. Berry, Classical Elasticity Theory [in Russian], GIFML, Moscow (1961).
7. L. V. Kantorovich and G. P. Akilov, Functional Analysis [in Russian], Nauka, Moscow (1977).